Objectives:

- Review inverse functions and define inverse trigonometric functions.
- Use implicit differentiation to find derivatives of inverse trigonometric functions.

Review of Inverse Functions:

To find the inverse of a function, reverse the roles of input and output.
If $f(a) = b$ then $\underline{\quad f^{-1}(b) = a \quad}$.

If $f(a) = f(c) = b$ for some $a \neq c$, we have a problem—what should $f^{-1}(b)$ be?

So, a function is only invertible if each input has a unique output. If a function has this nice quality, we say it is $\underline{\quad \text{one-to-one} \quad}$.

Graphically, a function is invertible if it passes the $\underline{\quad \text{horizontal line} \quad}$ test.

If f is invertible and $f(a) = b$, then $f^{-1}\big(f(a)\big) = \underline{\quad f^{-1}(b) = a \quad}$ and $f\big(f^{-1}(b)\big) = \underline{\quad f(a) = b \quad}$.

Inverse Trigonometric Functions:

Let's say we want to find an inverse function for $f(x) = \sin(x)$,

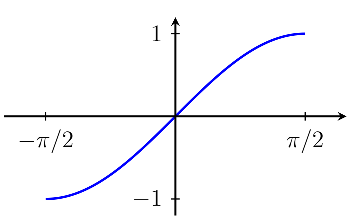
which we will call $\underline{\quad f^{-1}(x) = \arcsin(x) \quad}$.

First, we need to restrict the domain of $\sin(x)$ so that we have an invertible function:

The domain we choose is $\underline{\quad [-\pi/2, \pi/2] \quad}$.

Note: Why choose this domain?
There are other sections of the domain of $\sin(x)$ we could choose $\Big($e.g. $[\pi/2, 3\pi/2]$ or $[101\pi/2, 103\pi/2]\Big)$, but the convention is to choose $[-\pi/2, \pi/2]$ so we all agree on what we're talking about.

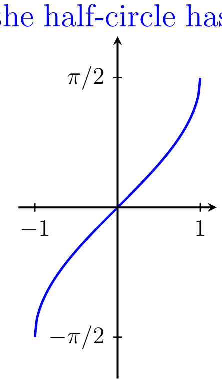
We are choosing the right half of the unit circle so each point on the half-circle has a different y-value.



$\sin(x)$, restricted

Domain: $[-\pi/2, \pi/2]$

Range: $[-1, 1]$



$\arcsin(x)$

Domain: $[-1, 1]$

Range: $[-\pi/2, \pi/2]$

Another important way of understanding this function is that $y = \arcsin(x)$ means:

1. $\sin(y) = x$

   AND

2. y is between $-\pi/2$ and $\pi/2$

Now we have a new function! How can we use it?

What should the angles be in a right triangle with a hypotenuse of 3 cm and one side that is $\sqrt{3}$ cm? If θ is the angle opposite the side of length $\sqrt{3}$, $\sin(\theta) = \frac{\sqrt{3}}{3}$. Since $-1 \leq \frac{\sqrt{3}}{3} \leq 1$, we can use $\arcsin\left(\sqrt{3}/3\right) = \theta$ to find what θ should be. This is not a special angle, so using a calculator $\theta = \arcsin\left(\sqrt{3}/3\right) \approx 0.615 \approx 35.26°$. Then the third angle must be about $90° - 35.26° \approx 54.74°$

Suppose the position of a particle is given by $s(t) = \sin(t)$. When is the particle at position $\frac{1}{2}$?

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$, so at $t = \frac{\pi}{6}$, the particle is at position $\frac{1}{2}$. Since $s(t)$ is periodic, we also know $s(t) = \frac{1}{2}$ at $t = \frac{\pi}{6} + 2\pi, \frac{\pi}{6} - 2\pi, \frac{\pi}{6} + 4\pi$, etc. Further, $\sin(t) = \frac{1}{2}$ also at $\frac{5\pi}{6}$. So we have another set of solutions at $t = \frac{5\pi}{6} + 2\pi, \frac{5\pi}{6} - 2\pi, \frac{5\pi}{6} + 4\pi \ldots$, etc. All of these solutions can be described by $\frac{\pi}{6} + 2\pi n$ and $\frac{5\pi}{6} + 2\pi m$, where n and m are integers.

The derivative of $\arcsin(x)$**:**
To find the derivative of $\arcsin(x)$, we're going to use our sneaky technique of implicit differentiation:

First: Rewrite the equation for $\arcsin(x)$ in terms of $\sin(x)$:

$$y = \arcsin(x)$$

$$\sin(y) = x$$

Second: Differentiate both sides:

$$\frac{d}{dx}\big(\sin(y)\big) = \frac{d}{dx}(x)$$

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

Third: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Fourth: Substitute to find a way to express $\dfrac{dy}{dx}$ in terms of x.

Notice that:

$$\big(\sin(y)\big)^2 + \big(\cos(y)\big)^2 = 1$$
$$\big(\cos(y)\big)^2 = 1 - \big(\sin(y)\big)^2$$
$$\cos(y) = \pm\sqrt{1 - \big(\sin(y)\big)^2}$$

On the interval we care about, $y \in [-\pi/2, \pi/2]$, $\cos(y)$ is always positive, so

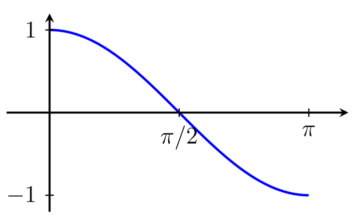
$$\cos(y) = \sqrt{1 - \big(\sin(y)\big)^2}$$

So,

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - \big(\sin(y)\big)^2}} = \frac{1}{\sqrt{1-x^2}}$$

In today's activity you'll see another way to use implicit differentiation to find the derivative of a trigonometric function.

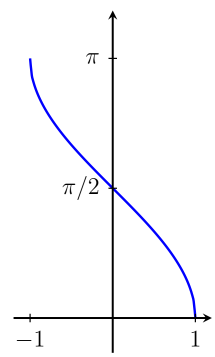
The last inverse trigonometric function we'll see is $\arccos(x)$:



$\cos(x)$, restricted

Domain: $[0, \pi]$

Range: $[-1, 1]$



$\arccos(x)$

Domain: $[-1, 1]$

Range: $[0, \pi]$

To find the derivative, we'll use implicit differentiation again.

$$y = \arccos(x)$$
$$\cos(y) = x$$
$$\frac{d}{dx}\big(\cos(y)\big) = \frac{d}{dx}(x)$$
$$-\sin(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(x)}} = \frac{-1}{\sqrt{1-x^2}}$$

Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\big(\arccos(x)\big) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\big(\arctan(x)\big) = \frac{1}{1+x^2}$$

Examples:

(a) Find the derivative of $f(x) = \arctan\left(x^3\right)$.

$$f'(x) = \frac{1}{1+\left(x^3\right)^2}\left(3x^2\right) = \frac{3x^2}{1+x^6}$$

(b) $y = x^2 e^{\arcsin(x)}$. Find y'.

$$y' = x^2\left(e^{\arcsin(x)}\right)' + (2x)e^{\arcsin(x)} = x^2\left(e^{\arcsin(x)}\right)\left(\frac{1}{1-x^2}\right) + (2x)e^{\arcsin(x)}$$